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HAMILTONIAN CIRCUITS IN POLYTOPES WITH EVEN SIDED FACES

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ABSTRACT
It has been conjectured that if P is a simple 3-polytope all of whose faces have an even number of sides, then P has a Hamiltonian circuit. In this paper it is shown that, if all the faces of P are either quadrilaterals or hexagons, then P does have a Hamiltonian circuit.

1. Introduction

It is well known that there are simple 3-polytopes (planar 3-valent, 3-connected graphs, see [4] or [2]) which do not have Hamiltonian circuits, see for example Tutte [5] or Faulkner and Younger [1]. But all the known examples have at least one face which is a $(2n+1)$-gon for some $n = 1, 2, \cdots$. So it is natural to consider those simple 3-polytopes all of whose faces are $2n$-gons, $n = 1, 2, \cdots$ and, in fact, it has been conjectured by D. Barnette [6] that they all admit Hamiltonian circuits. At present there appear to be no known results concerning this problem. A natural starting point is to investigate those simple 3-polytopes all of whose faces are quadrilaterals or hexagons. It is known (see [2]) that this is an infinite family of polytopes and we shall denote it by $\mathcal{P}$. The purpose of this work is to show that every polytope in $\mathcal{P}$ admits a Hamiltonian circuit, thus verifying one of the conjectures in [3].

I should like to thank Professor J. Malkevitch for drawing my attention to this problem, and Professor B. Grünbaum for his many helpful suggestions concerning an earlier form of this paper.

2. The results

Let $\mathcal{Q}$ denote the family of simple 3-polytopes with seven quadrilateral faces,

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one octagonal face with an adjacent quadrilateral, and any number of hexagonal faces. Then we shall prove the following

THEOREM. *The graph of each polytope P in $\mathcal{P} \cup \mathcal{Q}$ admits a Hamiltonian circuit. Moreover, if $P \in \mathcal{P}$, then for each chosen edge common to a quadrilateral and a hexagon there is a Hamiltonian circuit containing that edge; if $P \in \mathcal{Q}$, a Hamiltonian circuit may be found that passes through any preassigned edge common to the octagon and a quadrilateral.*

The proof of the theorem may be carried out by induction on the number of faces in P. It is clear that the statement is correct when P is a cube or hexagonal prism. If P is a polytope in $\mathcal{P} \cup \mathcal{Q}$ with ten or more faces and e is the chosen edge in P, we shall construct a polytope P^* in $\mathcal{P} \cup \mathcal{Q}$ (with fewer faces) having an edge e^* suitably situated. Then we shall show that from each Hamiltonian circuit of P^* containing e^* we obtain a Hamiltonian circuit of P containing e. Several cases have to be considered and we shall list them diagrammatically. In each case one has to show that P^* is in $\mathcal{P} \cup \mathcal{Q}$ and that from each circuit of P^* containing e^* the appropriate circuit of P can be obtained. It transpires that the latter task is trivial in each case, whilst the difficulty of the former varies from case to case. We shall indicate the types of arguments used by working through one of the cases in detail.

Case 1. One vertex of the quadrilateral with the chosen edge e belongs to a *clump* of three quadrilaterals with a common vertex (Fig. 1). We note that if $P \in \mathcal{P}$, then P consists of two clumps of quadrilaterals spaced by rings of hexagons and thus, given any appropriate edge e of P, there is a Hamiltonian circuit passing through e. So we can assume $P \in \mathcal{Q}$.

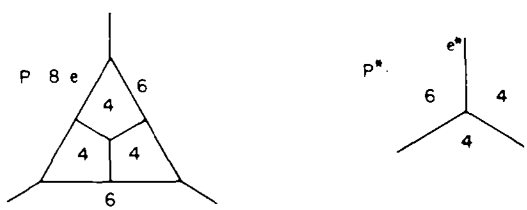

Fig. 1

Case 2. The quadrilateral with the chosen edge e belongs to a *patch* of three quadrilaterals, only two of which are not adjacent (Fig. 2, 3, 4, & 5). We note that if $P \in \mathcal{P}$, then P contains two such patches spaced by pairs of hexagons and thus, given any appropriate edge e of P, there is a Hamiltonian circuit passing through e. So we can assume $P \in \mathcal{Q}$. Further, if P contains four

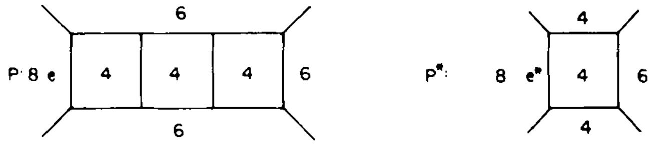

Fig. 2

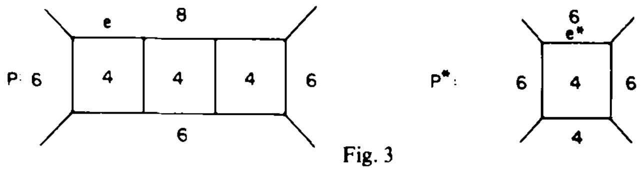

Fig. 3

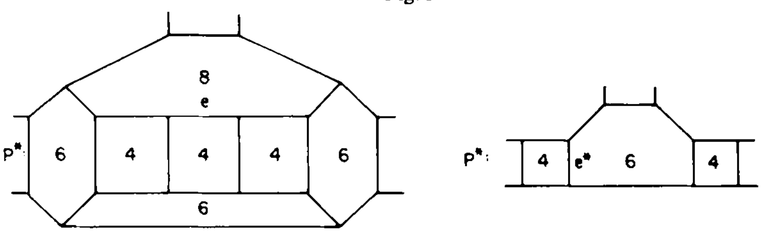

Fig. 4

quadrilaterals in a row adjacent to the octagon (Fig. 5), then, given any appropriate edge e of P, each Hamiltonian circuit of P^* will generate the required circuit of P.

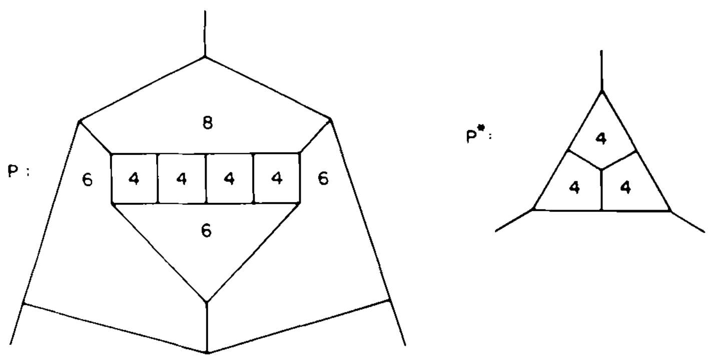

Fig. 5

Case 3. The quadrilateral with the chosen edge e has one adjacent quadrilateral but does not belong to a patch of three quadrilaterals (Figs. 6 & 7).

Case 4. The quadrilateral with the chosen edge e has no adjacent quadrilaterals (Fig. 8).

To indicate the method of proof we shall deal with the case illustrated in Fig. 8 with $P \in \mathcal{P}$. Showing that $P^* \in \mathcal{P} \cup \mathcal{Q}$ clearly amounts to proving that P^* is 3-connected. We do this by using the following results ([2], p. 272).

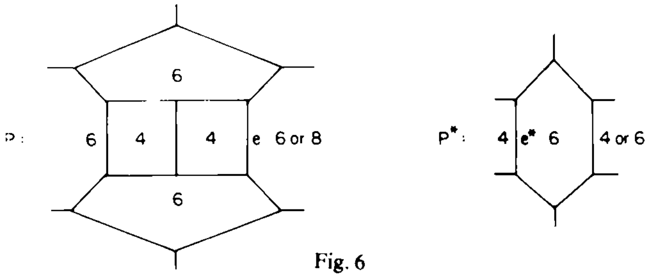

Fig. 6

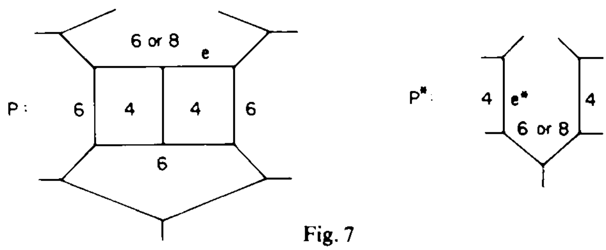

Fig. 7

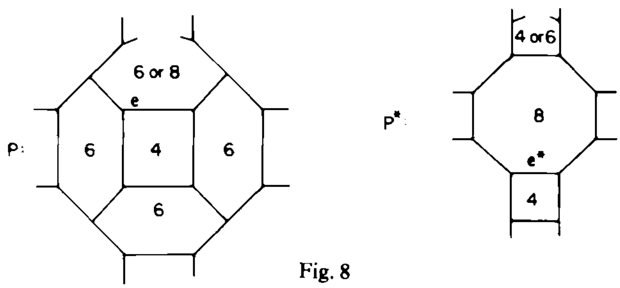

Fig. 8

LEMMA. (i) *Any connected* 3-*valent graph, all of whose faces have an even number of sides, is* 2-*connected.*

(ii) *There are no connected* 3-*valent graphs all of whose faces, apart from one, have an even number of sides.*

(iii) *There are no connected* 3-*valent graphs all of whose faces, apart from two adjacent ones, have an even number of sides.*

Thus we see, from (i), that P^* is 2-connected. We now use the notation of Fig. 9. If P^* is not 3-connected, then two of F_8, F_9, F_{10}, F_6, F_{13}, F_{12} must coincide. If $F_6 = F_8$, then this face has two edges in common with F_5. If $F_6 = F_9$, then this face must be hexagonal and so, since P is 3-connected, F_7 would have to be a triangle. If $F_6 = F_{10}$, this must be hexagonal and then F_8 and F_{12} must be triangles. So we can assume that $F_9 = F_{13}$. If F_i, F_j (G_i, G_j) are adjacent edges of P (P^*), we denote their common edge by f_{ij} (g_{ij}). We can clearly assume that the face $F_9 = F_{13}$ is hexagonal and that it is the unbounded

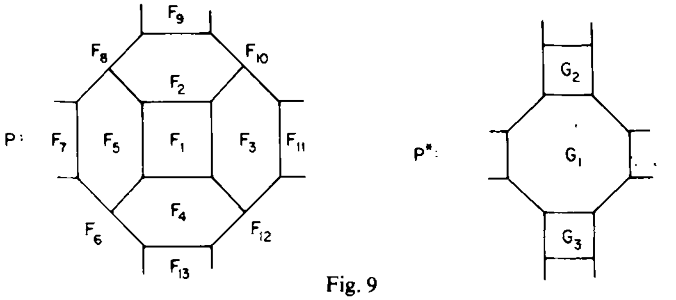

Fig. 9

face in Fig. 9. Then it has f_{69}, f_{89}, f_{29} as three consecutive edges. We now erase all the edges lying wholly in the interior of the region bounded by the edges f_{69}, f_{89}, f_{28}, f_{58}, f_{57}, f_{56}, f_{46}.

Replace F_7 by a quadrilateral F'_7, create a new triangular face F_{14} adjacent to F'_7 on the side of F'_7 "opposite to F_5" and having its third vertex on f_{68}. Then F_6, F_8 become F'_6, F'_8, both being hexagonal. We have thus created a graph contradicting assertion (ii) of the lemma. So we deduce that $P^* \in \mathcal{P} \cup \mathcal{Q}$.

Finally, we show that any circuit of P^* containing g_{13} ($= e^*$) gives a circuit of P containing f_{12}($= e$). If the circuit of P^* also contains g_{12}, we replace g_{13} by f_{45}, f_{14}, f_{34} and g_{12} by f_{25}, f_{12}, f_{23}. Otherwise we replace g_{13} by f_{45}, f_{15}, f_{12}, f_{13}, f_{34}. In each case we obtain the required circuit of P.

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